Multi-Product Price Optimization and Competition under the Nested Logit Model with Product-Dependent Price Sensitivities

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“red-bus/blue-bus” paradox

- Airport transportation: taxi or bus with attraction \( a_t = a_b \).
  \[
  \Pr(\text{bus}) = \Pr(\text{taxi}) = \frac{a_b}{a_t + a_b} = \frac{1}{2}.
  \]

- Two buses: red/blue with \( a_{rb} = a_{bb} = a_t \).
  - MNL model:
    \[
    \Pr(\text{red bus}) = \frac{a_{rb}}{a_{rb} + a_{bb} + a_t} = \frac{1}{3}.
    \]
  - Nested Logit (NL) model:
    - Stage II:
      \[
      \Pr(\text{red bus}|\text{bus}) = \frac{a_{rb}}{a_{rb} + a_{bb}} = \frac{1}{2}.
      \]
    - Stage I:
      \[
      \Pr(\text{red bus}) = \Pr(\text{bus}) \cdot \Pr(\text{red bus}|\text{bus}).
      \]
      If \( \Pr(\text{bus}) = \frac{1}{2} \), then \( \Pr(\text{red bus}) = \frac{1}{4} \).
IIA property: the ratio of probabilities of choosing any two alternatives is independent of the availability and attributes of a third alternative. In reality, adding new alternative reduces the probability of choosing similar alternatives more than dissimilar alternatives.

- MNL model:
  - Suffer from IIA property, not realistic
  - Often rejected by data

- NL model
  - Two level structure
  - Alleviate IIA property
  - Better fit to data
 Nested Logit model

- Two stage structure
  - Upper stage: select nest $i$ with probability $Q_i$
  - Lower stage: select a product $k$ within nest $i$ with $q_k|_i$.

- Nested Logit model:

$$Q_i(p_i, p_{-i}) = \frac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}},$$

$$q_k|_i(p_i) = \frac{e^{\alpha_{ik} - \beta_{ik} p_{ik}}}{\sum_{s=1}^{m_i} e^{\alpha_{is} - \beta_{is} p_{is}}},$$

$$\pi_{ik}(p_i, p_{-i}) = Q_i(p_i, p_{-i}) \cdot q_k|_i(p_i),$$

where $I_l = \log \sum_{s=1}^{m_l} e^{\alpha_{ls} - \beta_{ls} p_{ls}}$ represents the attractiveness of nest $l$, $\gamma_l$ refers to degree of inter-nest heterogeneity.
Optimization w.r.t. Market-Shares

Firm $i$’s total expected profit:

$$R_i(p_i, p_{-i}) \overset{\text{def}}{=} \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \pi_{ik}(p_i, p_{-i}).$$

Price $p_{ik}$ can be expressed in terms of $\pi_i := (\pi_{i1}, \pi_{i2}, \ldots, \pi_{im_i})$ as follows

$$p_{ik}(\pi_i) = \frac{1}{\beta_{ik}} (\log Q_i - \log \pi_{ik}) + \frac{1}{\beta_{ik} \gamma_i} (\log(1 - Q_i) - \log Q_i) + \frac{\alpha_{ik}}{\beta_{ik}} - \frac{\log(1 + a_{-i})}{\beta_{ik} \gamma_i}.$$ 

The profit of firm $i$ can be rewritten as a function of market-shares as follows

$$R_i(\pi_i, p_{-i}) = \sum_{k=1}^{m_i} \left( \frac{1}{\beta_{ik}} (\log Q_i - \log \pi_{ik}) + \frac{1}{\beta_{ik} \gamma_i} (\log(1 - Q_i) - \log Q_i) - \tilde{c}_{ik} \right) \cdot \pi_{ik}.$$ 

Profit function $R_i(\pi_i, p_{-i})$ is NOT concave in market shares.
Firm $i$’s total expected profit:

$$R_i(p_i, p_{-i}) \overset{\text{def}}{=} \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \pi_{ik}(p_i, p_{-i}).$$  \hspace{1cm} (1)$$

First order condition with $p_{ij}$,

$$\frac{\partial R_i(p_i, p_{-i})}{\partial p_{ij}} = \pi_{ij}(p_i, p_{-i}) \cdot \left[ 1 - \beta_{ij}(p_{ij} - c_{ij}) + \beta_{ij} (1 - \gamma_i(1 - Q_i(p_i, p_{-i}))) \cdot \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(p_i) \right] = 0,$$

satisfied by either $\pi_{ij}(p_i, p_{-i}) = 0$, requiring $p_{ij} = \infty$, or

$$1 - \beta_{ij}(p_{ij} - c_{ij}) + \beta_{ij} (1 - \gamma_i(1 - Q_i(p_i, p_{-i}))) \cdot \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(p_i) = 0.$$
Optimization w.r.t. Prices, cont.

The interior solution is equivalent to:

$$p_{ij} - c_{ij} - \frac{1}{\beta_{ij}} = \left(1 - \gamma_i \left(1 - Q_i(p_i, p_{-i})\right)\right) \sum_{s \in F_i} (p_{is} - c_{is}) q_s |_i(p_i).$$  \hspace{1cm} (2)

Let $F_i$ be the set of products whose prices satisfy (2) for all $j \in F_i$; $p_{ij} = \infty$ for $j \notin F_i$.

Proposition

The profit function is strictly monotone increasing in $F_i$, so it is optimal to offer all products at prices satisfying (2).

Observation

*Adjusted markup* $p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$ is constant for all $j$, denoted by $\theta_i$. 
The IIA Property and the Nested Logit Model
Optimizing the Prices of a Single Nest
Optimizing Prices over all the Nests
Applications to Oligopoly and Dynamic Pricing

**Nested Logit model: single-variable optimization**

Nest price price optimization (1) can be simplified to an optimization problem over a single decision variable:

\[ R_i(\theta_i, \theta_{-i}) = Q_i(\theta_i, \theta_{-i})(\theta_i + w_i(\theta_i)) \]  

where

\[
q_{k|i}(\theta_i) = \frac{e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i}}{\sum_{s \in F_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}},
\]

\[ Q_i(\theta_i, \theta_{-i}) = \frac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}}, \]

\[ \pi_{ik}(\theta_i, \theta_{-i}) = q_{k|i}(\theta_i) \cdot Q_i(\theta_i, \theta_{-i}), \]

\[ w_i(\theta_i) = \sum_{k \in F_i} \frac{1}{\beta_{ik}} \cdot q_{k|i}(\theta_i). \]

**Proposition**

(a) Function \( R_i(\theta_i, \theta_{-i}) \) is strictly unimodal with respect to \( \theta_i \) if \( \gamma_i \geq 1 \) or

\[
\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1 - \gamma_i}. \]

(b) The optimal \( \theta_i^* \) is in a bounded interval.
If condition $\gamma_i \geq 1$ or $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$ is not satisfied, then $R_i(\theta_i, \theta_i)$ is not unimodal with respect to $\theta_i$.

**Figure:** Non-unimodality of $R_i(\theta_i, \theta_i)$
Maximize total expected profit,

$$\max_{p_1, \ldots, p_n} R(p) \overset{\text{def}}{=} \sum_{i=1}^{n} \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \pi_{ik}(p_i, p_{-i}).$$  \hspace{1cm} (4)

Simplify to maximize \textit{adjusted markups} $\theta = (\theta_1, \ldots, \theta_n)$:

$$\max_{\theta_1, \ldots, \theta_n} R(\theta) \overset{\text{def}}{=} \sum_{i=1}^{n} Q_i(\theta_i, \theta_{-i})(\theta_i + w_i(\theta_i)).$$  \hspace{1cm} (5)
Proposition

The adjusted nested markup $\theta_i + (1 - \frac{1}{\gamma_j}) w_i(\theta_i)$, called adjusted nested markup, is constant for all $i$, denoted as $\phi$.

Thus, problem (5) can be reduced to an optimization problem with respect to single variable $\phi$,

$$
\max_{\phi} \quad R(\phi) \overset{\text{def}}{=} \sum_{i=1}^{n} Q_i(\theta_i, \theta_i) (\theta_i + w_i(\theta_i)),
$$

where $\theta_i + (1 - \frac{1}{\gamma_i}) w_i(\theta_i) = \phi, \forall i = 1, 2, \ldots, n$. 

Proposition

Function $R(\phi)$ is strictly unimodual in $\phi$ if $\gamma_i \geq 1$ or $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$ for all $i$. 

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Optimization and Competition under General Nested Logit Model
Oligopolistic Competition

- Each firm controls a nest of products
- Simultaneously choose prices
- Demand follows Nested Logit model

**Game I:**

\[ R_i(p_i, p_{-i}) = \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \cdot \pi_{ik}(p_i, p_{-i}) \]

**Game II:**

\[ R_i(\theta_i, \theta_{-i}) = Q_i(\theta_i, \theta_{-i})(\theta_i + w_i(\theta_i)) \]

**Observation**

Game I and Game II are equivalent. Function \( R_i(\theta_i, \theta_{-i}) \) is log-separable.

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Consider the derivatives of $\log R_i(\theta_i, \theta_{-i})$:

\[
\frac{\partial \log R_i(\theta_i, \theta_{-i})}{\partial \theta_i} = -\gamma_i (1 - Q_i(\theta_i, \theta_{-i}))v_i(\theta_i) + \frac{w_i(\theta_i)v_i(\theta_i)}{\theta_i + w_i(\theta_i)},
\]

\[
\frac{\partial^2 \log R_i(\theta_i, \theta_{-i})}{\partial \theta_i \partial \theta_j} = \gamma_i \gamma_j Q_i(\theta_i, \theta_{-i})Q_j(\theta_j, \theta_{-j})v_i(\theta_i)v_j(\theta_j) \geq 0, \quad \forall j \neq i.
\]

**Proposition**

**Game II** is a log-supermodular game; the equilibrium set has the componentwise largest and smallest elements. The largest equilibrium $\bar{\theta}^*$ is preferred by all the firms.
Oligopolistic Competition: symmetric game

Symmetric game: parameters \((\alpha_i, \beta_i, \gamma_i)\) in the NL model and the cost vector \(c_i\) are the same for each firm \(i = 1, 2, \ldots, n\).

**Proposition**

(a) Only symmetric equilibria exist for this symmetric game.
(b) The equilibrium is unique if \(\gamma_i \geq \frac{n}{n-1}\) or \(\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1 - \frac{n-1}{n}} \cdot \gamma_i\) for each \(i\).
Oligopolistic Competition (General): unique equilibrium

**Tatonnement Process**: Select a feasible vector $\theta^{(0)}$; in the $k^{th}$ iteration determine the optimal response for each firm $i$ as follows:

$$\theta_i^{(k)} = \arg \max R_i(\theta_i, \theta_{-i}^{(k-1)})$$

(7)

**Proposition**

Under mild conditions, **Game II** has a unique pure Nash equilibrium, denoted by $\theta^*$. It can be computed by the tatonnement scheme.
Example

Consider an example with two firms and each firm sells two products. \( \alpha_1 = (1.0, 2.0) \), \( \beta_1 = (0.6, 0.8) \) and \( \gamma_1 = 0.75 \); \( \alpha_2 = (0.8, 1.1) \), \( \beta_2 = (0.7, 1.2) \) and \( \gamma_2 = 0.5 \).

Figure: Convergence of Tatonnement Process
Extension: Nested Attraction Model

- $a_{is}(p_{is})$: the attractiveness of product $s$ of firm $i$ at price $p_{is}$
- $I_l = \log \sum_{s=1}^{m_l} a_{ls}(p_{is})$: the total attractiveness of firm $i$
- Probability:

\[
Q_i(p_i, p_{-i}) = \frac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}},
\]

\[
q_{k|i}(p_i) = \frac{a_{ik}(p_{ik})}{\sum_{s=1}^{m_i} a_{is}(p_{is})}.
\]
Proposition

Under some mild conditions, \((p_{ij} - c_{ij}) + \frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}\) is constant at optimal prices for all \(j\), denoted by \(\eta_i\).

Price optimization simplifies to one variable per-nest.

\[
\max_{\eta_i} R_i(\eta_i, \eta_{-i}) \overset{\text{def}}{=} Q_i(p_i, p_{-i}) \cdot \sum_{k=1}^{m_i} (p_{ik} - c_{ik})q_{k|i}(p_i),
\]

where \((p_{ij} - c_{ij}) + \frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})} = \eta_i\).

Corollary

The results apply for the Nested linear attraction model and the Nested modified CES model.
Application: dynamic pricing

- Products consume a common resource
- $x$ denote the remaining units of capacity
- $V(t, x)$ is the expected revenue-to-go function starting at state $(t, x)$

The Bellman equation is the following,

$$V(t, x) = \lambda_t \left\{ \max_p \sum_{k=1}^{m} (p_k - \Delta V(t - 1, x)) \cdot \pi_k(p) \right\} + V(t - 1, x),$$

where $\Delta V(t, x) = V(t, x) - V(t, x - 1)$ and

$$\pi_k(p) = Q(p) \cdot q_k(p) = \frac{\left( \sum_{s=1}^{m} e^{\alpha_s - \beta_s p_s} \right)^\gamma}{1 + \left( \sum_{s=1}^{m} e^{\alpha_s - \beta_s p_s} \right)} \cdot \frac{e^{\alpha_k - \beta_k p_k}}{\sum_{s=1}^{m} e^{\alpha_s - \beta_s p_s}}.$$
Let $r(\rho)$ be the maximum revenue rate subject to the aggregate consumption at rate $\rho$:

$$
r(\rho) \overset{def}{=} \max_{p} \sum_{k=1}^{m} p_k \pi_k(p) \\
\text{s.t.,} \quad \sum_{k=1}^{m} \pi_k(p) = \rho.
$$

Proposition

Function $r(\rho)$ is concave in $\rho$ if $\gamma_i \geq 1$ or $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$.

Then, the Bellman equation can be rewritten as follows:

$$
V(t, x) = \lambda_t \max_{0 \leq \rho \leq 1} \{r(\rho) - \rho \Delta V(t - 1, x)\} + V(t - 1, x).
$$
Conclusion

- Nested model alleviates IIA property.
- Adjusted markup is constant, adjusted nested markup is constant.
- Simplify to single-dimensional problem.
- Oligopolistic competition.
- Application in dynamic pricing
- Nested general attraction model
Thank you!